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Analytic Bethe ansatz for 1D Hubbard model and twisted coupled *XY* model

Ruihong Yue† and Tetsuo Deguchi
Department of Physics, Faculty of Science, Ochanomizu University, 112 Tokyo, Japan

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Abstract. We have found the eigenvalues of the transfer matrices for the 1D Hubbard model and for the coupled XY model with a twisted boundary condition by using the analytic Bethe ansatz method. Under a particular condition the two models have the same Bethe ansatz equations. We have also proved that the periodic 1D Hubbard model is exactly equal to the coupled XY model with a non-trivial twisted boundary condition at the level of Hamiltonians and transfer matrices.

1. Introduction

The 1D Hubbard model (the Hubbard model) is one of the significant exactly solvable models in condensed matter physics. Lieb and Wu [1] succeeded in diagonalizing the Hamiltonian in the frame of the coordinate Bethe ansatz. However, the integrability of the Hubbard model was recently set up by Shastry [2, 3], Olmedilla and co-workers [4–6] from the viewpoint of the quantum inverse scattering method (QISM). The Yang–Baxter equation for the related R-matrix was proved in [7]. In [2, 3], Shastry found the Yang–Baxter relation and constructed the transfer matrix of the coupled XY model which is equal to the Hubbard model with the help of the Jordan–Wigner transformation. The commutative family with one free parameter ensures the integrability of the system. Using a different approach, Olmedilla and co-workers [4–6], starting from the super-L-operator of the Hubbard model, solved the super-Yang–Baxter (SYB) relation and found the invertible R-matrix, which is the same as that given by Shastry [8] up to a scalar function. In both cases [3, 5], the Hamiltonian can be derived from the transfer matrix under a periodic boundary condition.

The basis of the quantum inverse scattering method is the Yang–Baxter relation. The latter is closely related to the quantum group and the Yangian. This technique provides a systematic method for dealing with integrable 1D quantum systems and 2D solvable statistical mechanical models. It is known that most integrable systems can be handled in the frame of the algebraic Bethe ansatz (or analytic Bethe ansatz) method. However, up to now there has been no report about the diagonalization of the transfer matrix for the Hubbard model in the QISM approach. Thus, it is important to find the solution of the Hubbard model by using QISM. In [8], Shastry conjectured the eigenvalue of the transfer matrix of the coupled XY model based upon the coordinate Bethe ansatz method. Comparing the Bethe ansatz equations given by Lieb and Wu [1] and by Shastry [8], an extra factor appears and it shows some difference between the two models. Furthermore, an

† E-mail: yue@sofia.phys.ocha.ac.jp

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exact calculation of the eigenvalues of the conserved quantities of the 1D Hubbard model has not yet been given. We recall that existence of infinite number of conserved quantities is fundamental to the integrability of the 1D Hubbard model [2, 3, 6].

The motivation behind this paper is to find the eigenvalues of the transfer matrices related to the Hubbard model and the coupled XY model with a twisted boundary condition by a version of analytic Bethe ansatz method (ABA). The systematic derivation of the eigenvalues of the transfer matrices provides an exact method for calculating the eigenvalues of the conserved quantities. We also discuss the difference between the coupled XY and the Hubbard models. The extra factor originates from the boundary condition. In fact, the Jordan–Wigner transformation does not maintain the invariance of the boundary condition. We show that the Hubbard model with a periodic boundary condition is exactly equal to the coupled XY model with a special boundary condition.

The organization of this paper is as follows. We recall the Yang–Baxter relation for the Hubbard model and the parametrization of the R-matrix in section 2. In section 3, we investigate special cases of the model by using the algebraic Bethe ansatz method. Because there are two kinds of creation operators with spin-up (or spin-down), such as T_{21} and T_{43} (T_{21} and T_{43}), the general multiparticle states become very complicated. However, the special solution gives an insight into the general structure of the eigenvalue. In section 4 we will apply an analytic Bethe ansatz method to the Hubbard model. In our approach, the analytic property of the eigenvalue of the transfer matrix, together with the asymptotic behaviour, determines almost all the unknown functions. The discussion is different from the standard analytic Bethe ansatz method, since there is no crossing symmetry, which played an important role in the standard method [9–12]. In section 5, we will apply the analytic Bethe ansatz method developed in section 4 to the coupled XY model with a twisted boundary condition. Under a special choice, it recovers the results of the Hubbard model at the level of the Bethe ansatz equations and the eigenvalue. In section 6 we show that the Hubbard model with a periodic boundary condition is equal to the coupled XY model with a special boundary condition by using the Jordan–Wigner transformation. Some discussions are given in section 7.

2. The Hubbard model and the super-Yang–Baxter relation

In this section we recall the definition of the SYB relation for the Hubbard model and some useful functional relations. We follow the notation of [5]. The Hamiltonian of the Hubbard model is

$$H_{\mathrm{Hu}} = -\sum_{m=1,s=\uparrow,\downarrow}^{L} \left(a^{+}_{m+1,s}a_{m,s} + a^{+}_{m,s}a_{m+1,s}\right) + U\sum_{m=1}^{L}(n_{m\uparrow} - \tfrac{1}{2})(n_{m\downarrow} - \tfrac{1}{2}) \tag{1}$$

where $a^{+}_{m,s}$ ($a_{m,s}$) stands for the mth site electron creation (annihilation) operator with spin s. The super-Yang–Baxter relation [5] is

$$\mathcal{R}(\mu,\nu)\,[\mathcal{L}_m(\mu)\otimes_s \mathcal{L}_m(\nu)] = [\mathcal{L}_m(\nu)\otimes_s \mathcal{L}_m(\mu)]\,\mathcal{R}(\mu,\nu). \tag{2}$$

The super tensor product is defined by

$$[A\otimes_s B]^{ia}_{jb} = A^{i}_{j}B^{a}_{b}(-1)^{[p(i)+p(j)]p(a)} \tag{3}$$

where $p(1) = p(4) = 0$, $p(2) = p(3) = 1$ are the parities. For the Hubbard model the L-operator takes the form

$$\mathcal{L}_m(\mu) = \begin{pmatrix} -e^{h(\mu)} f_{m\uparrow} f_{m\downarrow} & f_{m\uparrow} a_{\downarrow} & i a_{m\uparrow} f_{m\downarrow} & i a_{m\uparrow} a_{m\downarrow} e^{h(\mu)} \\ -i f_{m\uparrow} a^{+}_{m\downarrow} & f_{m\uparrow} g_{m\downarrow} e^{-h(\mu)} & a_{m\uparrow} a^{+}_{m\downarrow} e^{-h(\mu)} & i a_{m\uparrow} g_{m\downarrow} \\ a^{+}_{m\uparrow} f_{m\downarrow} & a^{+}_{m\uparrow} a_{m\downarrow} e^{-h(\mu)} & g_{m\uparrow} f_{m\downarrow} e^{-h(\mu)} & g_{m\uparrow} a_{m\downarrow} \\ -i a^{+}_{m\uparrow} a^{+}_{m\downarrow} e^{h(\mu)} & a^{+}_{m\uparrow} g_{m\downarrow} & i g_{m\uparrow} a^{+}_{m\downarrow} & -g_{m\uparrow} g_{m\downarrow} e^{h(\mu)} \end{pmatrix} \tag{4}$$

where μ is the spectrum parameter, and the functions f_{ms} and g_{ms} are

$$\begin{aligned} f_{ms} &= w_4(\mu) - w_3(\mu) - \{w_4(\mu) - w_3(\mu) - i[w_4(\mu)) + w_3(\mu)]\} n_{ms} \\ g_{ms} &= w_4(\mu) + w_3(\mu) - \{w_4(\mu) + w_3(\mu) - i[w_4(\mu) - w_3(\mu)]\} n_{ms} \\ \alpha(\mu) &= w_4(\mu) + w_3(\mu) = \sin(\mu + \pi/4) \\ \gamma(\mu) &= w_4(\mu) - w_3(\mu) = \sin(\mu - \pi/4) \\ \sinh(2h(\mu)) &= -\tfrac{1}{4} U \cos(2\mu). \end{aligned} \tag{5}$$

The graded $\mathcal{R}$-matrix reads

$$\begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & i\rho_6 & 0 & 0 & \rho_8 & 0 & 0 & 0 \\
0 & \frac{\rho_{10}}{i} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\rho_6 & 0 & 0 & \rho_5 & 0 & 0 & \rho_7 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \frac{\rho_{10}}{i} & 0 & 0 \\
0 & 0 & \frac{\rho_{10}}{i} & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & \rho_7 & 0 & 0 & \rho_5 & 0 & 0 & i\rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & \frac{\rho_{10}}{i} & 0 \\
0 & 0 & 0 & \rho_8 & 0 & 0 & i\rho_6 & 0 & 0 & \frac{\rho_6}{i} & 0 & 0 & \rho_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & i\rho_9 & 0 & 0 & \rho_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix} \tag{6}$$

where all functions are defined by

$$\begin{aligned}
\rho_1(\mu,\nu) &= e^l\alpha(\mu)\alpha(\nu) + e^{-l}\gamma(\mu)\gamma(\nu) \\
\rho_4(\mu,\nu) &= e^l\gamma(\mu)\gamma(\nu) + e^{-l}\alpha(\mu)\alpha(\nu) \\
\rho_9(\mu,\nu) &= -e^l\alpha(\mu)\gamma(\nu) + e^{-l}\gamma(\mu)\alpha(\nu) \\
\rho_{10}(\mu,\nu) &= e^l\gamma(\mu)\alpha(\nu) - e^{-l}\alpha(\mu)\gamma(\nu) \\
\rho_3(\mu,\nu) &= \frac{e^l\alpha(\mu)\alpha(\nu) - e^{-l}\gamma(\mu)\gamma(\nu)}{\alpha^2(\mu)-\gamma^2(\nu)} \\
\rho_5(\mu,\nu) &= \frac{-e^l\gamma(\mu)\gamma(\nu) + e^{-l}\alpha(\mu)\alpha(\nu)}{\alpha^2(\mu)-\gamma^2(\nu)} \\
\rho_6(\mu,\nu) &= \frac{e^{-h}[e^l\alpha(\mu)\gamma(\mu) - e^{-l}\alpha(\nu)\gamma(\nu)]}{\alpha^2(\mu)-\gamma^2(\nu)} \\
\rho_7(\mu,\nu) &= \rho_4(\mu,\nu) - \rho_5(\mu,\nu) \\
\rho_8(\mu,\nu) &= \rho_1(\mu,\nu) - \rho_3(\mu,\nu) \\
l &= h(\mu) - h(\nu) \\
h &= h(\mu) + h(\nu)\,.
\end{aligned} \tag{7}$$

Due to the super-Yang–Baxter relation, one can define the monodromy matrix

$$\mathcal{T}(\mu) = \mathcal{L}_L(\mu)\cdots\mathcal{L}_2(\mu)\mathcal{L}_1(\mu) \tag{8}$$

which still satisfies the super-Yang–Baxter relation

$$\mathcal{R}(\mu,\nu)\,[\mathcal{T}(\mu)\otimes_s\mathcal{T}(\nu)] = [\mathcal{T}(\nu)\otimes_s\mathcal{T}(\mu)]\,\mathcal{R}(\mu,\nu). \tag{9}$$

The super-Yang–Baxter relation leads to the existence of the commutative family of transfer matrices $t_H(\mu) = str\mathcal{T}(\mu)$ with infinitely many different values of μ. Thus, the Hubbard model is integrable. The infinite number of conserved quantities can be derived from the transfer matrix $t_H(\mu)$. The derivation of $\log[t_H(\mu)]$ at $\mu = \pi/4$ gives the Hamiltonian of the Hubbard model with a periodic boundary condition. Before ending this section, we list some useful functional relations which will be used in the following sections:

$$\begin{aligned}
1 &= \rho_1\rho_4 + \rho_9\rho_{10} \\
2 &= \rho_1\rho_5 + \rho_3\rho_4 \\
1 &= \rho_3\rho_5 - (\rho_6)^2 \\
\rho_{10} &= \rho_6(e^h\alpha(\mu)\alpha(\nu) + e^{-h}\gamma(\mu)\gamma(\nu)) \\
\rho_9 &= \rho_6(e^{-h}\alpha(\mu)\alpha(\nu) + e^h\gamma(\mu)\gamma(\nu))\,.
\end{aligned} \tag{10}$$

3. An algebraic analysis of the eigenvalue

In this section, we will discuss the solution of the Hubbard model in some special cases by using the algebraic Bethe ansatz method. For the general multiparticle states, it becomes very difficult even if there are two electrons with opposite spins.

Taking the special elements of the super-Yang–Baxter relation, one can obtain the following useful relations:

$$\mathcal{T}_{44}(\mu)\mathcal{T}_{43}(\nu) = \frac{\mathcal{R}_{44}^{44}(\nu,\mu)}{\mathcal{R}_{34}^{43}(\nu,\mu)}\mathcal{T}_{43}(\nu)\mathcal{T}_{44}(\mu) - \frac{\mathcal{R}_{34}^{34}(\nu,\mu)}{\mathcal{R}_{34}^{43}(\nu,\mu)}\mathcal{T}_{43}(\mu)\mathcal{T}_{44}(\nu) \tag{11}$$

$$\mathcal{T}_{33}(\mu)\mathcal{T}_{43}(\nu) = -\frac{\mathcal{R}_{33}^{33}(\mu,\nu)}{\mathcal{R}_{34}^{43}(\mu,\nu)}\mathcal{T}_{43}(\nu)\mathcal{T}_{33}(\mu) + \frac{\mathcal{R}_{43}^{43}(\mu,\nu)}{\mathcal{R}_{34}^{43}(\mu,\nu)}\mathcal{T}_{43}(\mu)\mathcal{T}_{33}(\nu) \tag{12}$$

$$\begin{aligned}\mathcal{T}_{22}(\mu)\mathcal{T}_{43}(\nu) = &-\left(\frac{\mathcal{R}_{23}^{32}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)} - \frac{\mathcal{R}_{23}^{41}(\mu,\nu)\mathcal{R}_{14}^{32}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)\mathcal{R}_{14}^{41}(\mu,\nu)}\right)\mathcal{T}_{43}(\nu)\mathcal{T}_{22}(\mu)\\ &+\left(\frac{\mathcal{R}_{23}^{14}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)} - \frac{\mathcal{R}_{23}^{41}(\mu,\nu)\mathcal{R}_{14}^{14}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)\mathcal{R}_{14}^{41}(\mu,\nu)}\right)\mathcal{T}_{41}(\nu)\mathcal{T}_{24}(\mu)\\ &-\left(\frac{\mathcal{R}_{23}^{23}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)} - \frac{\mathcal{R}_{23}^{41}(\mu,\nu)\mathcal{R}_{14}^{23}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)\mathcal{R}_{14}^{41}(\mu,\nu)}\right)\mathcal{T}_{42}(\nu)\mathcal{T}_{23}(\mu)\\ &+\frac{\mathcal{R}_{23}^{41}(\mu,\nu)\mathcal{R}_{42}^{42}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{41}(\mu)\mathcal{T}_{24}(\nu) + \frac{\mathcal{R}_{23}^{41}(\mu,\nu)}{\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{21}(\mu)\mathcal{T}_{44}(\nu)\\ &+\frac{\mathcal{R}_{42}^{42}(\mu,\nu)}{\mathcal{R}_{24}^{42}(\mu,\nu)}\mathcal{T}_{42}(\mu)\mathcal{T}_{23}(\nu)\end{aligned} \tag{13}$$

$$\begin{aligned}T_{11}(\mu)\mathcal{T}_{43}(\nu) = &\frac{\mathcal{R}_{13}^{31}(\mu,\nu)}{\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{43}(\nu)\mathcal{T}_{11}(\mu) + \frac{\mathcal{R}_{13}^{13}(\mu,\nu)}{\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{41}(\nu)\mathcal{T}_{13}(\mu)\\ &+\frac{\mathcal{R}_{23}^{41}(\mu,\nu)}{\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{21}(\mu)\mathcal{T}_{33}(\nu) + \frac{\mathcal{R}_{32}^{41}(\mu,\nu)}{\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{31}(\mu)\mathcal{T}_{23}(\nu)\\ &-\frac{\mathcal{R}_{41}^{41}(\mu,\nu)}{\mathcal{R}_{14}^{41}(\mu,\nu)}\mathcal{T}_{41}(\mu)\mathcal{T}_{13}(\nu).\end{aligned} \tag{14}$$

For the Hubbard model the Hilbert space consists of four states: a double occupied state $|\uparrow\downarrow\rangle$, a spin-up state $|\uparrow\rangle$, a spin-down state $|\downarrow\rangle$ and an unoccupied state $|0\rangle$. We denote them by $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$, respectively. It is convenient to introduce the reference state

$$|\text{vac}\rangle = |4\rangle_1 \otimes_s \cdots \otimes_s |4\rangle_L. \tag{15}$$

Using the explicit expression for the L-operator, one can find that the monodromy matrix acting on the reference state takes the form

$$\mathcal{T}(\mu)|\text{vac}\rangle = \begin{pmatrix} A_1(\mu) & 0 & 0 & 0\\ \mathcal{T}_{21}(\mu) & A_2(\mu) & 0 & 0\\ \mathcal{T}_{31}(\mu) & 0 & A_3(\mu) & 0\\ \mathcal{T}_{41}(\mu) & \mathcal{T}_{42}(\mu) & \mathcal{T}_{43}(\mu) & A_4(\mu)\end{pmatrix}|\text{vac}\rangle \tag{16}$$

where

$$A_4(\mu) = [-\alpha^2(\mu)e^{h(\mu)}]^L$$

$$A_2(\mu) = A_3(\mu) = [\alpha(\mu)\gamma(\mu)e^{-h(\mu)}]^L$$

$$A_1(\mu) = [-\gamma^2(\mu)e^{h(\mu)}]^L.$$

Unlike the case where the nested Bethe ansatz method is applicable, the operator $\mathcal{T}_{21}$ is related to the operator $\mathcal{T}_{43}$, creating an electron with spin-down. Generally, the relation is very complicated. When they act on the reference state, however, the situation becomes simple. An algebraic calculation shows that

$$\mathcal{T}_{21}(\tilde{\mu})|\mathrm{vac}\rangle = \frac{(-1)^L \gamma^{2L-1}(\tilde{\mu}) e^{(L-1)h(\tilde{\mu})}}{\gamma^{L-1}\alpha(\mu)^L e^{-(L-1)h(\mu)}} \mathcal{T}_{43}(\mu)|rac\rangle \tag{17}$$

where $\tilde{\mu}$ is defined by

$$e^{-2h(\tilde{\mu})} \frac{\alpha(\tilde{\mu})}{\gamma(\tilde{\mu})} = e^{2h(\mu)} \frac{\alpha(\mu)}{\gamma(\mu)}. \tag{18}$$

Furthermore, one can show that

$$\mathcal{T}_{21}(\tilde{\mu})\mathcal{T}_{43}(\mu_1)\cdots\mathcal{T}_{43}(\mu_n)|\mathrm{vac}\rangle \propto \mathcal{T}_{43}(\mu)\mathcal{T}_{43}(\mu_1)\cdots\mathcal{T}_{43}(\mu_n)|\mathrm{vac}\rangle\,.$$

It is worth noting that this relation is valid only for the reference state; when acting on the other states, it will be invalid. There is a similar relation between $\mathcal{T}_{31}$ and $\mathcal{T}_{42}$. Due to this relation, we can construct the special states for all spin-up (spin-down) with $\mathcal{T}_{43}$ ($\mathcal{T}_{42}$):

$$|\Psi_N\rangle = \mathcal{T}_{43}(\mu_1)\cdots\mathcal{T}_{43}(\mu_N)|\mathrm{vac}\rangle.$$

Using the commutative relations, we can find

$$t_H(\mu)|\Psi_N\rangle = \Lambda(\mu)|\Psi_N\rangle + \text{unwanted \ terms}.$$

where

$$\Lambda(\mu) = A_4(\mu)\prod_{j=1}^{N}\frac{\rho_1(\mu_j,\mu)}{i\rho_9(\mu_j,\mu)} - A_3(\mu)\prod_{j=1}^{N}\frac{-\rho_4(\mu,\mu_j)}{i\rho_9(\mu,\mu_j)} - A_2(\mu)\prod_{j=1}^{N}\frac{-i\rho_{10}(\mu,\mu_j)}{\rho_1(\mu,\mu_j)-\rho_3(\mu,\mu_j)}$$

$$+A_1(\mu)\prod_{j=1}^{N}\frac{-i\rho_{10}(\mu,\mu_j)}{\rho_1(\mu,\mu_j)-\rho_3(\mu,\mu_j)}. \tag{19}$$

The vanishing of the unwanted terms gives the Bethe ansatz equation

$$\left[-e^{2h(\mu_j)}\frac{\alpha(\mu_j)}{\gamma(\mu_j)}\right]^L = 1. \tag{20}$$

Thus, the states $|\Psi_N\rangle$ are really the eigenstates of the transfer matrix $t(\mu)$ if the spectrum parameters are appropriately chosen to satisfy the Bethe ansatz equation (20).

The general states with N spin-down and $N-M$ spin-up are given by sums of products of the combination of $\mathcal{T}_{21}$, $\mathcal{T}_{31}$, and $\mathcal{T}_{4j}$, $j=1,2,3$ acting on the reference state. Let us consider a special case $N-M=M=1$. The general form is

$$|\Psi_{1,1}\rangle = \{f_1(\mu_1,\mu_2)\mathcal{T}_{42}(\mu_1)\mathcal{T}_{43}(\mu_2) + f_2(\mu_1,\mu_2)\mathcal{T}_{43}(\mu_1)\mathcal{T}_{42}(\mu_2)$$

$$+\ f_3(\mu_1,\mu_2)\mathcal{T}_{21}(\mu_1)\mathcal{T}_{31}(\mu_2) + f_4(\mu_1,\mu_2)\mathcal{T}_{31}(\mu_1)\mathcal{T}_{21}(\mu_2)$$

$$+\ f_5(\mu_1,\mu_2)\mathcal{T}_{41}(\mu_1) + f_6(\mu_1,\mu_2)\mathcal{T}_{41}(\mu_2)\}\,|\mathrm{vac}\rangle. \tag{21}$$

We can determine the all coefficients f_j by requiring the right-hand side of (21) to be the eigenvector of the transfer matrix. The general solution is very complicated. Fortunately, we can show by an explicit calculation that $|\Psi_{1,1}\rangle$ is the eigenstate if $f_3 = f_4 = 0$ with appropriate f_5 and f_6. Using the super-Yang–Baxter relation, we find the eigenvalue for the case when $f_1 = -f_2$:

$$\Lambda(\mu) = A_4(\mu) \prod_{j=1}^{2} \frac{\rho_1(\mu_j, \mu)}{i\rho_9(\mu_j, \mu)} + A_1(\mu) \prod_{j=1}^{N} \frac{-i\rho_{10}(\mu, \mu_j)}{\rho_1(\mu, \mu_j) - \rho_3(\mu, \mu_j)}$$

$$-A_2(\mu)\Bigg\{ \frac{\rho_4(\mu, \mu_1)\rho_{10}(\mu, \mu_2)}{\rho_9(\mu, \mu_1)[\rho_1(\mu, \mu_2) - \rho_3(\mu, \mu_2)]}$$

$$+ \frac{\rho_4(\mu, \mu_2)\rho_{10}(\mu, \mu_1)}{\rho_9(\mu, \mu_2)[\rho_1(\mu, \mu_1) - \rho_3(\mu, \mu_1)]}$$

$$-\left[\frac{\rho_{10}(\mu, \mu_1)}{\rho_1(\mu, \mu_1) - \rho_3(\mu, \mu_1)} + \frac{\rho_4(\mu, \mu_1)}{\rho_9(\mu, \mu_1)} \right]$$

$$\times \left[\frac{\rho_{10}(\mu, \mu_2)}{\rho_1(\mu, \mu_2) - \rho_3(\mu, \mu_2)} + \frac{\rho_4(\mu, \mu_2)}{\rho_9(\mu, \mu_2)} \right] \Bigg\}. \tag{22}$$

The vanishing of the unwanted terms gives the Bethe ansatz equation:

$$\left[\prod_{j=1}^{2} \frac{-\alpha(\mu_j)}{\gamma(\mu_j)} e^{2h(\mu_j)} \right]^L = 1. \tag{23}$$

For the case $f_1 = f_2$ the eigenvalue and the Bethe ansatz equations are given by (19) and (20), respectively. It is worth pointing out that these states considered here are not complete. For example, if one considers the case $f_1 = f_2 = 0$, one can get similar results.

4. Analytic Bethe ansatz for the Hubbard model

In section 3, we applied the algebraic Bethe ansatz method to some eigenstates of the Hubbard model. For general states, the previously straightforward calculation becomes very complicated. In this section, however, we want to discuss an analytic Bethe ansatz method for the same problem based on the hints given by the above results. We should generalize the standard ABA [9–12] in which the crossing symmetry and asymptotic behaviour play a key role. We cannot apply the same argument to the eigenvalue of the Hubbard model in which there is no such crossing symmetry for the R-matrix.

Let us first investigate the special solutions and show how to generalize the method from the viewpoint of the ABA. One may understand that the analytic property of the function $\Lambda(\mu)$ leads to the Bethe ansatz equation (20). It is clear that $\rho_9(\mu, \mu_j) = 0$ is the simple pole of $\Lambda(\mu)$ (equation (19)). In order to keep the analytic property of the eigenvalue, the residue at such a pole must be zero. The Bethe ansatz equation is quite simply the condition of a vanishing residue. Similarly the vanishing residue at the pole $\rho_1(\mu, \mu_j) = \rho_3(\mu, \mu_j)$ gives the same Bethe ansatz equation. This property can be generalized to all kinds of states with different particles. The eigenvalue function should be analytic and has only superficial simple poles. The vanishing residues at such poles will give the Bethe ansatz equations.

Let us discuss the general eigenvalue of the Hubbard model. The special solutions (19) and (22), together with some standard knowledge of algebraic Bethe ansatz, contain enough information about the general one. It consists of four terms which are proportional to $A_j(\mu)$, $j = 1, 2, 3, 4$, respectively. The terms involving A_1 and A_4 are dependent only on the total number of electrons. The other terms depend on both the total number of electrons N and the number of spin-up electrons M, as shown in section 3. Thus, the general eigenvalue should be

$$\Lambda(\mu) = A_4(\mu)\prod_{j=1}^{N}\frac{\rho_1(\mu_j,\mu)}{i\rho_9(\mu_j,\mu)} - A_3(\mu)\prod_{j=1}^{N}\frac{-\rho_4(\mu,\mu_j)}{i\rho_9(\mu_j,\mu)}\prod_{m=1}^{M}g_3(\mu,\lambda_m)$$
$$-A_2(\mu)\prod_{j=1}^{N}\frac{-i\rho_{10}(\mu,\mu_j)}{\rho_1(\mu,\mu_j)-\rho_3(\mu,\mu_j)}\prod_{m=1}^{M}g_2(\mu,\lambda_m)$$
$$+A_1(\mu)\prod_{j=1}^{N}\frac{-i\rho_{10}(\mu,\mu_j)}{\rho_1(\mu,\mu_j)-\rho_3(\mu,\mu_j)} \tag{24}$$

where g_2 and g_3 are unknown functions. The μ_j and λ_m are free parameters. N is the total number of electrons and M the number of spin-up electrons.

We now show how the analytic property of the eigenvalue restricts the unknown functions. First, $\Lambda(\mu)$ has two sets of poles related to the parameters μ_j. One (case A) is controlled by the null denominator of the first two terms on the right-hand side of (24). The other (case B) is from the last two terms. For case A, the position of the poles is determined by

$$e^{2h(\mu)}\frac{\alpha(\mu)}{\gamma(\mu)} = e^{2h(\mu_j)}\frac{\alpha(\mu_j)}{\gamma(\mu_j)}. \tag{25}$$

Due to the $i\pi$-period of $h(\mu)$, we can get $\mu = \mu_j$ in the region $0 \leqslant \mu_j \leqslant \pi$. At these poles, the functions $\rho_1(\mu,\mu_j) - \rho_3(\mu,\mu_j)$ and $\rho_{10}(\mu,\mu_j)$ also vanish; however, the ratio is finite. So, the singularity at these poles is dominated by the terms on the first line. The null residue requires that

$$\left[-e^{2h(\mu_j)}\frac{\alpha(\mu_j)}{\gamma(\mu_j)}\right]^L = \prod_{m=1}^{M}g_3(\mu_j,\lambda_m). \tag{26}$$

For case B, the position of the poles satisfies

$$0 = [e^{-h(\mu_j)+h(\mu)}\gamma(\mu_j)\alpha(\mu_j) - e^{h(\mu_j)-h(\mu)}\gamma(\mu)\alpha(\mu)]$$
$$\times[e^{h(\mu)+h(\mu_j)}\gamma(\mu)\alpha(\mu_j) - e^{-h(\mu)-h(\mu_j)}\alpha(\mu)\gamma(\mu_j)]. \tag{27}$$

The first part is equal to $\rho_9(\mu,\mu_j) = 0$. It is not a pole due to $\rho_{10}(\mu,\mu_j) = 0$ at this point. The real pole is located at $\tilde{\mu}_j$:

$$e^{-2h(\tilde{\mu}_j)}\frac{\alpha(\tilde{\mu}_j)}{\gamma(\tilde{\mu}_j)} = e^{2h(\mu_j)}\frac{\alpha(\mu_j)}{\gamma(\mu_j)}. \tag{28}$$

The vanishing residue at $\mu = \tilde{\mu}_j$ gives

$$\left[-e^{2h(\tilde{\mu}_j)}\frac{\gamma(\tilde{\mu}_j)}{\alpha(\tilde{\mu}_j)}\right]^L = \prod_{m=1}^{N}g_2(\tilde{\mu}_j,\lambda_m). \tag{29}$$

In order to keep the analytic property of the eigenvalue, equations (26) and (28) must be satisfied simultaneously, which leads to the following functional relation:

$$\prod_{m=1}^{M}g_3(\mu_j,\lambda_m) = \prod_{m=1}^{M}g_2^{-1}(\tilde{\mu}_j,\lambda_m). \tag{30}$$

We find it convenient to introduce the new variables k and k_j:

$$e^{ik} = -e^{2h(\mu)}\frac{\alpha(\mu)}{\gamma(\mu)} \qquad e^{ik_j} = -e^{2h(\mu_j)}\frac{\alpha(\mu_j)}{\gamma(\mu_j)}. \tag{31}$$

In terms of the new variables the relation between the quantities with and without tildes is very simple:

$$\sin(\tilde{k}_j) = \sin(k_j) + \tfrac{1}{2}iU. \tag{32}$$

At this stage, we need to know some properties of the unknown functions g_i which are already hidden in the special solution with $N = 2,\ M = 1$. In terms of k_j, equation (22) becomes

$$\Lambda(\mu) = \left(-\alpha^2(\mu)e^{h(\mu)}\right)^L \Lambda(k)$$

$$\begin{aligned}\Lambda(k) = \prod_{j=1}^{2}\frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j)}\\
- e^{-ikL}\prod_{j=1}^{N}\frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j)}\\
\times\frac{2i\sin(k) - i\sin(K_1) - i\sin(k_2) - U/2}{2i\sin(k) - i\sin(k_1) - i\sin(k_2) + U/2}\\
- e^{-ikL}\prod_{j=1}^{2}\frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j) + U/2}\\
\times\frac{2i\sin(k) - i\sin(K_1) - i\sin(k_2) + 3U/2}{2i\sin(k) - i\sin(k_1) - i\sin(k_2) + U/2}\\
- e^{-i(k+\tilde{k})L}\prod_{j=1}^{2}\frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j) + U/2}.\end{aligned} \tag{33}$$

It is clear that the eigenvalue has another simple pole $2\sin(k) = \sin(k_1) + \sin(k_2) + \frac{1}{2}iU$. The Bethe ansatz equation ensures the analytic property of the eigenvalue. This strongly suggests that the undetermined functions have simple poles of the form $\sin(k) = \text{constant}$. On the other hand, the analytic property of the eigenvalue requires that g_2 and g_3 must have the same poles. Therefore, the general form of $g_2(\mu,\lambda)$ and $g_3(\mu,\lambda)$ is

$$g_2(\mu,\lambda) = \frac{P_2(k,\lambda)}{i\sin(k) - \lambda + U/4} \qquad g_3(\mu,\lambda) = \frac{P_3(k,\lambda)}{i\sin(k) - \lambda + U/4} \tag{34}$$

where the function $P_2(k,\lambda)$ and $P_3(k,\lambda)$ are integral functions. The general form is $P_r(k,\lambda) = \sum_{n=0} a_n^r(k)(\lambda)^n$. Substituting this in (30), we find that

$$\begin{aligned}a_0^2(k) &= [i\sin(k) - U/4]a_1^2(k)\\
a_0^3(k) &= [i\sin(k) + 3U/4]a_1^3(k)\\
a_n^2(k) &= a_n^2(k) = 0 \qquad n \geqslant 2\\
a_1^3(k) &= [a_1^2(\hat{k})]^{-1}\end{aligned} \tag{35}$$

where $\hat{k}$ is defined by $\sin(\hat{k}) = \sin(k) - iU/2$. Moreover, the function $a_1^2(k)$ is analytic and has no zero in the complex plane. It will be fixed by the asymptotic behaviour of the transfer matrix. Let us assume that $U \leqslant 0$ and $\mu \longrightarrow -i\infty$: then the eigenvalue approaches

$$\Lambda(\mu) \longrightarrow e^{3L\infty}\{[a_1^2(\infty)]^M + [a_1^2(\infty)]^{-M}\}. \tag{36}$$

Comparing this with the asymptotic behaviour of $t(\mu) \longrightarrow e^{3L\infty}$, we obtain the result that $a_1^2(\infty)$ is a no-zero constant. Based upon the knowledge of analysis such as Liouville's theorem on integral functions, we arrive at $a_1^2(k)$ being a no-zero constant. A special form of $\Lambda(\mu)$ under $N = M = 1$ fixes this constant to be unity. Finally, we arrive at the final results

$$\Lambda(\mu) = \left(-\alpha^2(\mu)e^{h(\mu)}\right)^L \Lambda(k)$$

$$\begin{aligned}\Lambda(k) = \prod_{j=1}^{N} \frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j)} \\ - e^{-ikL}\prod_{j=1}^{N} \frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j)} \prod_{m=1}^{M} \frac{i\sin(k) - \lambda_m - U/4}{i\sin(k) - \lambda_m + U/4} \\ - e^{-ikL}\prod_{j=1}^{N} \frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j) + U/2} \prod_{m=1}^{M} \frac{i\sin(k) - \lambda_m + 3U/4}{i\sin(k) - \lambda_m + U/4} \\ - e^{-i(k+\tilde{k})L}\prod_{j=1}^{N} \frac{2i\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k) - i\sin(k_j) + U/2}.\end{aligned} \tag{37}$$

The parameters satisfy the following Bethe ansatz equations:

$$\begin{aligned} e^{ik_jL} &= \prod_{m=1}^{M} \frac{i\sin(k_j) - \lambda_m - U/4}{i\sin(k_j) - \lambda_m + U/4} \\ -\prod_{m=1}^{M} \frac{\lambda_r - \lambda_m - U/2}{\lambda_r - \lambda_m + U/2} &= \prod_{j=1}^{N} \frac{i\sin(k_j) - \lambda_r + U/4}{i\sin(k_j) - \lambda_m - U/4}. \end{aligned} \tag{38}$$

Differentiating $\log(\Lambda(\mu))$ at $\mu = \pi/4$ will give the energy of the Hubbard model, which coincides with that given in [1]:

$$E = \frac{UL}{4} - \frac{NU}{2} - \sum_{j=1}^{N} \cos(k_j). \tag{39}$$

In general, the higher derivatives of $\log \Lambda(\mu)$ at $\mu = \pi/4$ give the eigenvalues of the conserved quantities at higher orders. We have checked that equations (37) and (38) with $N = 2,\ M = 1$ coincide with the result obtained in section 3.

5. The ABA for the coupled *XY* model

In this section we investigate the eigenvalue of the transfer matrix of the coupled XY model with a twisted boundary condition.

The L-operator related to the coupled XY model [2, 3] is

$$L_m(\mu) = \begin{pmatrix} e^{h(\mu)}p_m^+q_m^+ & p_m^+\tau_m^- & \sigma_m^-q_m^+ & e^{h(\mu)}\sigma_m^-\tau_m^- \\ p_m^+\tau_m^+ & e^{-h(\mu)}p_m^+q_m^- & e^{-h(\mu)}\sigma_m^-\tau_m^+ & \sigma_m^-q_m^- \\ \sigma_m^+q_m^+ & e^{-h(\mu)}\sigma_m^+\tau_m^- & e^{-h(\mu)}p_m^-q_m^+ & p_m^-\tau_m^- \\ e^{h(\mu)}\sigma_m^+\tau_m^+ & \sigma_m^+q_m^- & p_m^-\tau_m^+ & e^{h(\mu)}p_m^-q_m^- \end{pmatrix} \tag{40}$$

where σ_m^a and τ_m^a are two independent Pauli matrices located ath the mth site. The operators $p^{\pm}$ and $q^{\pm}$ read

$$\begin{aligned} p_m^{\pm} &= w_4(\mu) \pm w_3(\mu)\sigma_m^z \\ q_m^{\pm} &= w_4(\mu) \pm w_3(\mu)\tau_m^z\,. \end{aligned} \tag{41}$$

In [2, 3] it was shown that this L-operator satisfies the Yang–Baxter equation:

$$R(\mu,\nu)L_m(\mu) \otimes L_m(\nu) = L_m(\nu) \otimes L_m(\mu)R(\mu,\nu). \tag{42}$$

The R-matrix is

$$\begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & \rho_6 & 0 & 0 & \rho_6 & 0 & 0 & -\rho_8 & 0 & 0 & 0 \\
0 & \rho_{10} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \rho_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_6 & 0 & 0 & \rho_5 & 0 & 0 & -\rho_7 & 0 & 0 & \rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \rho_{10} & 0 & 0 \\
0 & 0 & \rho_{10} & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_6 & 0 & 0 & -\rho_7 & 0 & 0 & \rho_5 & 0 & 0 & \rho_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & \rho_{10} & 0 \\
0 & 0 & 0 & -\rho_8 & 0 & 0 & \rho_6 & 0 & 0 & \rho_6 & 0 & 0 & \rho_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & \rho_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1
\end{pmatrix}. \tag{43}$$

This Yang–Baxter relation ensures the monodromy matrix $T(\mu) = L_L(\mu) \otimes \cdots \otimes L_1(\mu)$ satisfying the Yang–Baxter relation. In order to simplify our calculation, we choose the ferromagnetic state (all states spin-down) as the reference state. From the explicit expression of the L-operator, we have

$$T(\mu)|\text{vac}\rangle = \begin{pmatrix} A_1(\mu) & 0 & 0 & 0 \\ T_{21}(\mu) & A_2(\mu) & 0 & 0 \\ T_{31}(\mu) & 0 & A_3(\mu) & 0 \\ T_{41}(\mu) & T_{42}(\mu) & T_{43}(\mu) & A_4(\mu) \end{pmatrix} |\text{vac}\rangle \tag{44}$$

where

$$\begin{aligned} A_4(\mu) &= [\alpha^2(\mu)e^{h(\mu)}]^L \\ A_2(\mu) &= A_3(\mu) = [\alpha(\mu)\gamma(\mu)e^{-h(\mu)}]^L \\ A_1(\mu) &= [\gamma^2(\mu)e^{h(\mu)}]^L. \end{aligned} \tag{45}$$

Similarly, one can define the transfer matrix $t(\mu) = stT(\mu)$ and find the eigenvalue of it, which will be related to the periodic boundary condition. The eigenvalue of the diagonal-of-diagonal transfer matrix of this model with a periodic condition was found by Bariev [13] in terms of the coordinate Bethe ansatz method.

In order to consider the twisted boundary condition, we introduce the generalized transfer matrix

$$t^g(\mu) = T_{11}(\mu)a^{i\beta_1} + T_{22}(\mu)a^{i\beta_2} + T_{33}(\mu)a^{i\beta_3} + T_{44}(\mu)a^{i\beta_4} \tag{46}$$

where

$$\begin{aligned} a^{i\beta_4} &= a^{-i\beta_1} = e^{a_\sigma N_\sigma + a_\tau N_\tau + a_0} \\ a^{i\beta_3} &= a^{-i\beta_2} = e^{c_\sigma N_\sigma + c_\tau N_\tau + c_0} \end{aligned} \tag{47}$$

where a_s and c_s are free parameters, N_σ (N_τ) the total number of σ-spin (τ-spin). Now, we want to find the eigenvalue of $t^g(\mu)$ by means of the analytic Bethe ansatz method. First, by using the algebraic Bethe ansatz method, we find the eigenvalue of the states with N τ-spin (or σ-spin) flipping from the reference state. After a long but direct calculation, we arrive at

$$\begin{aligned} \Lambda_N(\mu) &= e^{a_s N + a_0}[e^{h(\mu)}\alpha^2(\mu)]^L \prod_{j=1}^{N} \frac{\rho_1(\mu_j, \mu)}{\rho_9(\mu_j, \mu)} + e^{c_s N + c_0}[e^{-h(\mu)}\alpha(\mu)\gamma(\mu)]^L \prod_{j=1}^{N} \frac{\rho_4(\mu, \mu_j)}{\rho_9(\mu, \mu_j)} \\ &\quad + e^{-c_s N - c_0}[e^{-h(\mu)}\alpha(\mu)\gamma(\mu)]^L \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \mu_j)}{\rho_1(\mu_j, \mu) - \rho_3(\mu, \mu_j)} \\ &\quad + e^{-a_s N - a_0}[e^{h(\mu)}\gamma^2(\mu)]^L \prod_{j=1}^{N} \frac{\rho_1(\mu, \mu_j)}{\rho_3(\mu, \mu_j) - \rho_1(\mu, \mu_j)} \end{aligned} \tag{48}$$

where $s = \sigma$ ($s = \tau$) for all σ (τ) spin-up states, the parameters μ_j are determined by

$$\left[e^{2h(\mu_j)} \frac{\alpha(\mu_j)}{\gamma(\mu_j)} \right] = (-1)^{N+1} e^{(c_s - a_s)N + c_0 - a_0}. \tag{49}$$

From this expression and the similar arguments in section 4, we can write the general form of the eigenvalue as

$$\begin{aligned} \Lambda(\mu) &= e^{a_0 + a_\sigma M + a_\tau(N-M)}[e^{h(\mu)}\alpha^2(\mu)]^L \prod_{j=1}^{N} \frac{\rho_1(\mu_j, \mu)}{\rho_9(\mu_j, \mu)} + e^{c_0 + c_\sigma M + c_\tau(N-M)}[e^{-h(\mu)}\alpha(\mu)\gamma(\mu)]^L \\ &\quad \times \prod_{j=1}^{N} \frac{\rho_4(\mu, \mu_j)}{\rho_9(\mu, \mu_j)} \prod_{m=1}^{M} g_3(\mu, \lambda_m) + e^{-c_0 - c_\sigma M - c_\tau(N-M)}[e^{-h(\mu)}\alpha(\mu)\gamma(\mu)]^L \\ &\quad \times \prod_{j=1}^{N} \frac{\rho_{10}(\mu, \mu_j)}{\rho_1(\mu_j, \mu) - \rho_3(\mu, \mu_j)} \prod_{m=1}^{M} g_2(\mu, \lambda_m) \\ &\quad + e^{-a_0 - a_\sigma M - a_\tau(N-M)}[e^{h(\mu)}\gamma^2(\mu)]^L \prod_{j=1}^{N} \frac{\rho_1(\mu, \mu_j)}{\rho_3(\mu, \mu_j) - \rho_1(\mu, \mu_j)}. \end{aligned} \tag{50}$$

Second, consider the singularity of $\Lambda(\mu)$ at the poles related to the parameters μ_j. As was done in the Hubbard model, the null residue condition requires the following relation:

$$\prod_{m=1}^{M} g_3(\mu, \lambda_m) = \prod_{m=1}^{M} g_2^{-1}(\tilde{\mu}, \lambda_m). \tag{51}$$

This equation is same as that in the Hubbard model (equation (34)). So, we can use the results of the Hubbard model:

$$g_2(\mu,\lambda) = c\frac{i\sin(k)-\lambda-U/4}{i\sin(k)-\lambda+U/4}$$
$$g_3(\mu,\lambda) = \frac{1}{c}\frac{i\sin(k)-\lambda-U/4}{i\sin(k)-\lambda+U/4}. \qquad (52)$$

Here we have used the same definition of k as that used in the Hubbard model. One should note that in this case, the constant c in the above equation is not 1 as in the Hubbard model. Taking $N = M = 1$ in (50) and comparing with (48), one can get $c = -1$. Thus, we obtain the final results

$$\Lambda(\mu) = [e^{h(\mu)}\alpha^2(\mu)]^L\Lambda(k)$$

$$\begin{aligned}\Lambda(k) = {} & (-1)^N e^{a_0+a_\sigma M+a_\tau(N-M)}\prod_{j=1}^{N}\frac{2\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k)-i\sin(k_j)}\\ & + e^{c_0+c_\sigma M+c_\tau(N-M)}\prod_{j=1}^{N}\frac{2\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k)-i\sin(k_j)}\\ & \times(-1)^{L+M}e^{-ikL}\prod_{m=1}^{M}\frac{i\sin(k)-\lambda_m-U/4}{i\sin(k)-\lambda_m+U/4}\\ & + e^{-c_0-c_\sigma M-c_\tau(N-M)}\prod_{j=1}^{N}\frac{2\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k)-i\sin(k_j)+U/2}\\ & \times(-1)^{L+M+N}e^{-ikL}\prod_{m=1}^{M}\frac{i\sin(k)-\lambda_m+3U/4}{i\sin(k)-\lambda_m+U/4}\\ & + e^{-a_0-a_\sigma M-a_\tau(N-M)}e^{-i(k+\tilde{k})L}\prod_{j=1}^{N}\frac{2\cos((k+k_j)/2)\cos((\tilde{k}+k_j)/2)}{i\sin(k)-i\sin(k_j)+U/2}.\end{aligned} \qquad (53)$$

The Bethe ansatz equations are

$$(-1)^{M+N+1+L}e^{ik_jL} = e^{c_0-a_0+(c_\sigma-a_\sigma)M+(c_\tau-a_\tau)(N-M)}\prod_{m=1}^{M}\frac{i\sin(k_j)-\lambda_m-U/4}{i\sin(k_j)-\lambda_m+U/4}$$
$$\prod_{j=1}^{N}\frac{i\sin(k_j)-\lambda_r+U/4}{i\sin(k_j)-\lambda_r-U/4} = (-1)^{N+1}e^{2(c_0+c_\sigma M+c_\tau(N-M))}\prod_{m=1}^{M}\frac{\lambda_r-\lambda_m-U/2}{\lambda_r-\lambda_m+U/2}. \qquad (54)$$

These are the exact solution of the coupled XY model with twisted boundary condition. It is interesting that it recovers the results of the Hubbard model when $a_\sigma = a_\tau = c_\sigma = -c_\tau = -i\pi/2$ and $a_0 = iL\pi,\ c_0 = i\pi$. This gives a precise relation between the two models and make it clear why the extra factor appears in the Bethe ansatz equations of Lieb and Wu [1] and of Shastry [8]. When $a_s = c_s = 0,\ s = \sigma, \tau, 0$, they reduce to the periodic case. The correspondence between our notation and that in [8] is $2i\sin(k_j) = z_j^{-1} - z_j$.

6. The twisted boundary condition

In this section, we will discuss the boundary condition related to our generalized transfer matrix (46) and prove the equality of the periodic Hubbard model to the twisted coupled

XY model in terms of the transfer matrices and the Hamiltonians.

First, we derive the Hamiltonian related to the t^g, equation (46), by using the standard method. After a straightforward calculation, we arrive at

$$\begin{aligned} H = \sum_{m=1}^{L-1}(\sigma_{m+1}^+ a_m^- + \sigma_m^+\sigma_{m+1}^- + \tau_{m+1}^+\tau_m^- + \tau_m^+\tau_{m+1}^-) + \frac{U}{4}\sum_{m=1}^{L}\sigma_m^z\tau_m^z \\ + \exp\{-\epsilon[a_0 + c_0 + (a_\sigma + c_\sigma)N_\sigma + (a_\tau + c_\tau)N_\tau]\}\sigma_N^+\sigma_1^- \\ + \exp\{\epsilon[a_0 + c_0 + (a_\sigma + c_\sigma)N_\sigma + (a_\tau + c_\tau)N_\tau]\}\sigma_N^-\sigma_1^+ \\ + \exp\{-\epsilon'[a_0 - c_0 + (a_\sigma - c_\sigma)N_\sigma + (a_\tau - c_\tau)N_\tau]\}\tau_N^+\tau_1^- \\ + \exp\{\epsilon'[a_0 - c_0 + (a_\sigma - c_\sigma)N_\sigma + (a_\tau - c_\tau)N_\tau]\}\tau_N^-\tau_1^+ \end{aligned} \tag{55}$$

where

$$\epsilon = \begin{cases} 1 & \tau \text{ spin up} \\ -1 & \tau \text{ spin down} \end{cases} \qquad \epsilon' = \begin{cases} 1 & \sigma \text{ spin up} \\ -1 & \sigma \text{ spin down.} \end{cases} \tag{56}$$

This means that the Hamiltonian (55) gives the coupled XY model discussed in section 5 with a twisted boundary:

$$\begin{aligned} \sigma_{L+1}^\pm = e^{-\epsilon(a_\sigma + c_\sigma)}\exp\{\pm\epsilon[a_0 + c_0 + (a_\sigma + c_\sigma)N_\sigma + (a_\tau + c_\tau)N_\tau]\}\sigma_1^\pm \\ \tau_{L+1}^\pm = e^{-\epsilon'(a_\tau - c_\tau)}\exp\{\pm\epsilon[a_0 - c_0 + (a_\sigma - c_\sigma)N_\sigma + (a_\tau - c_\tau)N_\tau]\}\tau_1^\pm. \end{aligned} \tag{57}$$

Comparing the Bethe ansatz equations (37), (38), (53) and (54), we find that they are same if the free parameters are fixed:

$$a_\sigma = a_\tau = c_\sigma = -c_\tau = -\tfrac{1}{2}i\pi \qquad a_0 = iL\pi \qquad c_0 = i\pi. \tag{58}$$

Let us prove the connection (58) by using the Jordan–Wigner transformation on transfer matrix. The Jordan–Wigner transformation on operators is defined by

$$\begin{pmatrix}\sigma_m^+ \\ \sigma_m^-\end{pmatrix} = V_{m\uparrow}^2\begin{pmatrix}a_{m\uparrow}^+ \\ a_{m\uparrow}\end{pmatrix} = \begin{pmatrix} v_{m\uparrow}^2 & 0 \\ 0 & v_{m\uparrow}^{-2}\end{pmatrix}\begin{pmatrix}a_{m\uparrow}^+ \\ a_{m\uparrow}\end{pmatrix} \tag{59}$$

$$\begin{aligned} \begin{pmatrix}\tau_m^+ \\ \tau_{m\uparrow}^-\end{pmatrix} &= V_{m\downarrow}^2\begin{pmatrix}a_{m\downarrow}^+ \\ a_{m\downarrow}\end{pmatrix} \\ &= \begin{pmatrix} v_{m\uparrow}^2 u_{m\uparrow}^2 r_m^2 v_{m\downarrow}^2 & 0 \\ 0 & (v_{m\uparrow}u_{m\uparrow}r_m v_{m\downarrow})^{-2}\end{pmatrix}\begin{pmatrix}a_{m\downarrow}^+ \\ a_{m\downarrow}\end{pmatrix} \end{aligned} \tag{60}$$

with the definitions

$$\begin{aligned} v_{ms} &= \exp\left\{i\frac{\pi}{2}\sum_{k=1}^{m-1}(n_{ks} - 1)\right\} \\ u_{ms} &= \exp\left\{i\frac{\pi}{2}(n_{ks} - 1)\right\} \\ r_m &= \exp\left\{i\frac{\pi}{2}\sum_{k=m+1}^{L}(n_{k\uparrow} - 1)\right\}. \end{aligned} \tag{61}$$

Under the transformation, the L-operator becomes [5]:

$$\mathcal{L}_m(\mu) = V_{m+1}L_m(\mu)V_m^{-1} \tag{62}$$

where

$$
\begin{aligned}
V_{m+1} &= V_m(U_{m\uparrow} \otimes U_{m\uparrow}) \\
&= V_{m\uparrow}U_{m\uparrow} \otimes V_{m\downarrow}U_{m\downarrow} \qquad (63)\\
U_{m,s} &= \mathrm{dia}(u_{m,s}, u_{m,s}^{-1}).
\end{aligned}
$$

Substituting equation (62) in $t_H(\mu)$, we obtain

$$
\begin{aligned}
t_H(\mu) &= \mathcal{T}_{11}(\mu) - \mathcal{T}_{22}(\mu) - \mathcal{T}_{33}(\mu) + \mathcal{T}_{44}(\mu) \\
&= e^{\beta_1}T_{11}(\mu) + e^{\beta_2}T_{22}(\mu) + e^{\beta_3}T_{33}(\mu) + e^{\beta_4}T_{44}(\mu) \qquad (64)
\end{aligned}
$$

where

$$
\begin{aligned}
e^{\beta_1} &= \exp\left\{-i\frac{\pi}{4}\sum_{j=1}^{L}(\sigma_j^z + \tau_j^z - 2)\right\} = e^{-\beta_4} \\
e^{\beta_2} &= \exp\left\{-i\frac{\pi}{4}\sum_{j=1}^{L}(\sigma_j^z - \tau_j^z)\right\} = e^{-\beta_3}. \qquad (65)
\end{aligned}
$$

This is exactly equal to equation (46) with the condition (58). This completes our proof.

It is worth pointing out that Wadati *et al* [5] also applied the Jordan–Wigner transformation to the L-operator and the R-matrix. In the derivation of the Hamiltonian of the Hubbard model, they impose a periodic boundary condition. However, they did not consider the relation between the boundaries.

In the rest of this section, we give another independent proof in terms of the Hamiltonians. The periodic Hubbard model is

$$
\begin{aligned}
H = &- \sum_{m=1,s}^{L-1}(a_{m+1,s}^{+}a_{m,s} + a_{m,s}^{+}a_{m+1,s}) + \sum_{m=1,s}^{L}(n_{m\uparrow} - \tfrac{1}{2})(n_{m\downarrow} - \tfrac{1}{2}) \\
&- \sum_{s=\uparrow,\downarrow}(a_{1,s}^{+}a_{L,s} + a_{L,s}^{+}a_{1,s}) \qquad (66)
\end{aligned}
$$

where we have used the periodic condition

$$
a_{L+1,s}^{+} = a_{1,s}^{+} \qquad a_{L+1,s} = a_{1,s}. \qquad (67)
$$

Using the Jordan–Wigner transformation, we can obtain

$$
\begin{aligned}
a_{m+1,\uparrow}^{+}a_{m,\uparrow} + a_{m,\uparrow}^{+}a_{m+1,\uparrow} &= -(\sigma_{m+1}^{+}\sigma_m^{-} + \sigma_m^{+}\sigma_{m+1}^{-}) \\
a_{m+1,\downarrow}^{+}a_{m,\downarrow} + a_{m,\downarrow}^{+}a_{m+1,\downarrow} &= -(\tau_{m+1}^{+}\tau_m^{-} + \tau_m^{+}\tau_{m+1}^{-}) \\
a_{1,\uparrow}^{+}a_{L,\uparrow} + a_{L,\uparrow}^{+}a_{1,\uparrow} &= \exp\left\{\frac{i\pi}{2}\sum_{j=1}^{L}(\sigma_j^z - \tfrac{1}{2})\right\}(\sigma_L^{-}\sigma_1^{+} + \sigma_L^{+}\sigma_1^{-}) \qquad (68)\\
a_{1,\downarrow}^{+}a_{L,\downarrow} + a_{L,\downarrow}^{+}a_{1,\downarrow} &= \exp\left\{\frac{i\pi}{2}\sum_{j=1}^{L}(\tau_j^z - \tfrac{1}{2})\right\}(\tau_L^{-}\tau_1^{+} + \tau_L^{+}\tau_1^{-}).
\end{aligned}
$$

The Hamiltonian becomes

$$
\begin{aligned}
H = {} & \sum_{m=1}^{L-1}(\sigma_{m+1}^{+}\sigma_{m}^{-} + \sigma_{m+1}^{-}\sigma_{m}^{+}) - \exp\left\{\frac{i\pi}{2}\sum_{j=1}^{L}(\sigma_j^z - \tfrac{1}{2})\right\}(\sigma_L^{-}\sigma_1^{+} + \sigma_L^{+}\sigma_1^{-}) \\
& + \sum_{m=1}^{L-1}(\tau_{m+1}^{+}\tau_{m}^{-} + \tau_{m+1}^{-}\tau_{m}^{+}) - \exp\left\{\frac{i\pi}{2}\sum_{j=1}^{L}(\tau_j^z - \tfrac{1}{2})\right\}(\tau_L^{-}\tau_1^{+} + \tau_L^{+}\tau_1^{-}) \\
& + \frac{U}{4}\sum_{m=1}^{L}\sigma_m^z\tau_m^z.
\end{aligned}
\tag{69}
$$

Therefore, under the Jordan–Wigner transformation, the Hamiltonian of the Hubbard model becomes one of the coupled XY model with twisted boundary condition

$$
\begin{aligned}
\sigma_{L+1}^{\pm} &= \exp\left\{\pm\frac{i\pi}{2}\sum_{j=1}^{L}(\sigma_j^z - 1)\right\}\sigma_1^{\pm} \\
\tau_{L+1}^{\pm} &= \exp\left\{\pm\frac{i\pi}{2}\sum_{j=1}^{L}(\tau_j^z - 1)\right\}\tau_1^{\pm}
\end{aligned}
\tag{70}
$$

which is coincident with equations (57) and (58).

7. Concluding remarks

In this paper we have found the eigenvalues of the transfer matrices of the Hubbard model and the coupled XY model with a twisted boundary condition by using the analytic Bethe ansatz method. In fact, the matrix elements of the elements of L-operator, equations (4) and (40), can be interpreted as the Boltzmann weights in 2D statistical systems. The eigenvalues (37) and (53) are equivalent to the partition functions of the related 2D statistical models. The eigenvalues of the conserved quantities of the Hubbard model can be obtained exactly by taking the logarithmic derivative of the eigenvalues of the transfer matrix.

We have also shown how the 1D Hubbard model and the coupled XY model are equal at the levels of the Hamiltonian and the transfer matrix by using the Jordan–Wigner transformation. We recall that the power expansion of $\log(t^g(\mu))$ in terms of μ will give the infinite number of conserved quantities explicitly. At this level we claim that the coupled XY model with a twisted boundary condition is integrable.

It is worth pointing out that here we consider only a set of special boundary conditions. It is not difficult to generalize to other kinds of twisted boundary conditions. For the open boundary, one should consider the solution of the reflection equations. This will be related to the surface critical behaviour of the system.

The analytic Bethe ansatz discussed in the present paper can be applied to Shastry's inhomogeneous coupled six-vertex model. The partition functions and related Bethe ansatz were found. The results will be given in another paper.

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Note added. After finishing this paper, we were informed by Professor Wadati that Ramos and Martin [14] had found the eigenvalue of the Hubbard model by using the algebraic Bethe ansatz method, which partly recovers our results from a different approach. In [14], they also notice the effect of the boundary condition, but they do not discuss it in detail.

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